

On the Kobayashi-Royden pseudonorm
for almost complex manifolds

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August 15, 1997

Abstract

In this paper we define Kobayashi-Royden pseudonorm for almost complex manifolds. Its basic properties known from the complex analysis are preserved in the nonintegrable case as well. We prove that the pseudodistance induced by this pseudonorm coincides with the Kobayashi pseudodistance defined for the almost complex case earlier. We also consider a geometric application for moduli spaces of pseudoholomorphic curves.

Contents

Introduction	2
1 Definition of the pseudonorm and basic properties	3
2 Coincidence theorem	7
3 Connection with h -principle	9
4 Hyperbolicity and nonhyperbolicity	11
Bibliography	13

Introduction

In 1967 Kobayashi ([K1]) introduced biholomorphically-invariant pseudodistance (which is a distance without nondegenerace axiom) for any complex manifold. This gave rise for the theory of hyperbolic spaces (see [K2, La, PS] for details). The Kobayashi pseudodistance is the maximal pseudodistance among all nonincreasing under holomorphic maps pseudodistances and such that on the unit disk $\mathbf{D} \subset \mathbf{C}$ it coincides with the distance $d_{\mathbf{D}}$ which is induced by infinitesimal Poincaré metric of the Lobachevskii plane:

$$dl^2 = \frac{dzd\bar{z}}{(1 - |z|^2)^2}.$$

Another way to define the pseudodistance d_M on a complex manifold M is the following:

$$d_M(p, q) = \inf \sum_{k=1}^m d_{\mathbf{D}}(z_k, w_k),$$

where the infimum is taken over all holomorphic mappings $f_k : \mathbf{D} \rightarrow M$, $k = 1, \dots, m$, with the properties $f_1(z_1) = p$, $f_k(w_k) = f_{k+1}(z_{k+1})$ and $f_m(w_m) = q$. In the paper [KO] the Kobayashi pseudodistance notion was extended to the case of arbitrary almost complex manifolds and it was shown that basic properties of this pseudodistance stay preserved.

In 1970 Royden ([Ro]) discovered and justified an infinitesimal analog of the Kobayashi pseudodistance for complex manifolds. The main goal of the present paper is to define the corresponding notion in the category of almost complex manifolds and to prove the coincidence theorem. We also consider the reduction procedure which permits to define geometrical invariants for moduli spaces of pseudoholomorphic curves.*

* The author thanks all scientists from Math. and Stat. Dept. of the Univ. of Tromsøe where he was a visitor and where the paper was finished.

Chapter 1

Definition of the pseudonorm and basic properties

Let (M^{2n}, J) be an almost complex manifold, i.e. $J^2 = -\mathbf{1} \in T^*M \otimes TM$. We denote by $e = 1 \in T_0\mathbb{D}$ the unit vector. Let us also denote by \mathbb{D}_R the disc of radius R in \mathbb{C} . Let $v \in T_pM$, $p = \tau_M v$, where by $\tau_M : TM \rightarrow M$ we denote the canonical projection.

Definition. The Kobayashi-Royden pseudonorm is the function

$$F_M(v) = \inf_{\mathcal{R}} \frac{1}{r},$$

where the set \mathcal{R} consists of all pseudoholomorphic mappings $f : \mathbb{D} \rightarrow M$ (i.e. such mappings that $f_* \circ j_0 = J \circ f_*$) with the property $f_*(0)e = rv$, $r \in \mathbb{R}_+$.

By theorem III from [NW] the set $\mathcal{R} = \bigcup \mathcal{R}_r(v)$ is nonempty since $\mathcal{R}_r(v)$ is nonempty for every $r > 0$ small enough. Here $\mathcal{R}_r(v)$ stands for the set of all pseudoholomorphic curves f with $f_*(0)e = rv$, $r > 0$. Next statement is the direct consequence of the definition.

Proposition 1. *For any vector $v \in T_*(M_1)$ and for any pseudoholomorphic mapping $f : (M_1, J_1) \rightarrow (M_2, J_2)$ we have:*

$$F_{M_2}(f_*v) \leq F_{M_1}(v).$$

□

Proposition 2. (i). *The function F_M on TM is nonnegative and homogeneous of degree one: $F_M(tv) = |t|F_M(v)$ for any $v \in T_*M$ and $t \in \mathbb{R}$.*
(ii). *Let $K \subset M$ be a compact with nonempty set of interior points such that the almost complex structure J is tamed by an exact symplectic form $\omega = d\alpha$ (i.e. $\omega(\xi, J\xi) > 0$ for $\xi \neq 0$) on it. Then for any $v \in T_pM$ with $p = \tau_M v \in K$ and for any norm $|\cdot|$ on τ_M there exists a constant C_K such that*

$$F_M(v) \leq C_K |v|.$$

Proof. The first statement is obvious while the second is a reformulation of the nonlinear Schwarz lemma ([Gr1]). \square

Proposition 3. *The function F_M is upper semicontinuous.*

Proof. Our statement is equivalent to the following one: if there is a pseudoholomorphic disk $f : \mathbf{D}_R \rightarrow M$ in some direction $v \in T_*M$, $f_*e = v$, then in any close direction $v' \in \mathcal{V}(v)$ of v there is a pseudoholomorphic disk of almost the same size $f' : \mathbf{D}_{R-\varepsilon} \rightarrow M$, $(f')_*e = v'$. Here $\varepsilon > 0$ is arbitrary small fixed number. The last statement is equivalent to the existence theorem for some nonlinear partial differential equation. We prove it using the fixed point theorem in the Banach space following [NW]. This paper by theorem III provides existence of a small pseudoholomorphic disk in any given direction. The proof is based on the linearization of the almost complex structure at the point, on the writing down the corresponding nonlinear equation and on the proximity of the obtained equation to the linear equation on holomorphic curves of the linearized structure. In our case we can linearize the almost complex structure along given pseudoholomorphic curve $f : \mathbf{D}_R \rightarrow M$.

Contracting if needed we can assume that our manifold is a neighborhood of the disk $\mathbf{D}_R \subset \mathbf{C} \times \{0\}^{n-1} \subset \mathbf{C}^n$ and that the almost complex structure J coincides with the standard complex structure J_0 at the points of this disk. Let $k \in \mathbb{Z}_+$, $\lambda \in (0, 1)$. Denote by $C_{(R)}^{k+\lambda} = C^{k+\lambda}(\mathbf{D}_R; \mathbf{C}^n)$ the space of all λ -Hölder k -smooth mappings of \mathbf{D}_R into \mathbf{C}^n and consider this space equipped with the standard Hölder norm $\|\cdot\|$. Let's introduce also pseudonorm $\|f\|' = \max\{\|\partial f\|, \|\bar{\partial} f\|\}$. We use standard notations ∂_i and $\bar{\partial}_i$ defined regardless the complex structure J_0 . In this case the defining equation for pseudoholomorphic curve $f : \mathbf{D}_{R-\varepsilon}(\zeta) \rightarrow \mathbf{C}^n(z^i)$ in coordinates $z^i = z^i(\zeta)$ takes the form (see details in [NW]):

$$\bar{\partial} z^i + \sum_m a_{\bar{m}}^i(z) \bar{\partial} \bar{z}^m = 0, \quad z^i \in C^{k+\lambda}(\mathbf{D}_{R-\varepsilon}; \mathbf{C}^n). \quad (*)$$

Fixation of the point $p \in \mathbf{C}^n$ which the curve goes through and of the direction of the curve $v \in T_p M$ takes the form of initial conditions: $z^i(0) = p^i$, $\partial z^i(0) = v^i$. Due to our linearization we have $a_{\bar{m}}^i(z) = 0$ for all points $z \in \mathbf{D}_R \subset \mathbf{C}^n$. Thus we have the solution $z_0 = \zeta v_0$ in the direction of the vector $v_0 = (1, 0, \dots, 0) \in T_0 \mathbf{C}^n$, $|\zeta| < R$. We seek for a solution of the equation (*) at the form

$$z(\zeta) = p + \zeta v + \Theta(z, z),$$

$$\Theta^i(f, g)(\zeta) = \theta^i(f, g)(\zeta) - \theta^i(f, g)(0) - \zeta [\partial \theta^i(f, g)](0),$$

$$\theta^i(f, g) = -T \left(\sum_m a_{\bar{m}}^i(g) \bar{\partial} \bar{f}^m \right),$$

where $Tf(w) = \frac{1}{2\pi i} \int_{D_R} \frac{f(\zeta)}{\zeta - w} d\zeta \wedge d\bar{\zeta}$. Let us suppose that $a_{\bar{m}}^i \in C^{k+\lambda}$. Since the function z_0 is a solution of our equation then $\Theta(z_0, z_0) = 0$. Consider its neighborhood $\mathcal{O} \subset C_{(R-\varepsilon)}^{k+\lambda}$. Consider also a neighborhood \mathcal{V} of the vector v_0 in $T_* \mathbf{C}^n$. For sufficiently small neighborhood \mathcal{V} we seek a solution of the equation (*) in \mathcal{O} as the limit of the iteration process

$$z_1^{(v)} = p + \zeta v, \quad z_{k+1}^{(v)} = p + \zeta v + \Theta(z_k^{(v)}, z_k^{(v)}).$$

Lemma. For any given small $\varepsilon > 0$ the iterative process converges in $C_{(R-\varepsilon)}^{k+\lambda}$ for sufficiently small neighborhood \mathcal{V} and the limit is a function $z^{(v)} \in C_{(R-\varepsilon)}^{k+1+\lambda}$. This function is a solution of the equation (*) with initial conditions $z^{(v)}(0) = p$, $\partial z^{(v)}(0) = v$, and in addition $z^{(v)} \rightarrow z_0$ when $v \rightarrow v_0$ and for small enough \mathcal{V} we have: $\|z^{(v)}\|' \leq 2|v|$.

The required convergence is proved by estimates similar to [NW] but in our case the polidisk has the form $\mathbf{D}_{R-\varepsilon} \times (\mathbf{D}_{\delta(\varepsilon)})^{n-1} \subset \mathbf{C}^n$. The main estimate is the upper bound of the contraction coefficient of the process:

$$\|\Theta(f, f) - \Theta(g, g)\| < c \cdot |v - v_0|^\lambda \cdot \|f - g\|.$$

It is obtained via the following estimates (cf. [NW]):

1. $\|Th\|' \leq c_1 \cdot \|h\|.$
2. $\|\Theta(f, f) - \Theta(g, g)\| \leq \|\Theta(f, f) - \Theta(f, g)\| + \|\Theta(f, g) - \Theta(g, g)\|.$
3. $\|a_{\bar{m}}^i(f)\| \leq c_2 \cdot |v - v_0|^\lambda.$
4. $\|a_{\bar{m}}^i(f) - a_{\bar{m}}^i(g)\| \leq c_3 \cdot |v - v_0| \cdot \|f - g\|.$

□

Chapter 2

Coincidence theorem

Let us define the function $\bar{d}_M : M \times M \rightarrow \mathbb{R}$ by the formula

$$\bar{d}_M(p, q) = \inf_{\gamma} \int_0^1 F_M(\dot{\gamma}(t)) dt$$

where the infimum is taken over all piecewise smooth paths from the point p to the point q . The upper semicontinuity and upper boundness (propositions 2 and 3) gives us the correctness of this definition and

Proposition 4. *The function \bar{d}_M is a pseudodistance.* □

Theorem 1. *The pseudodistance just introduced coincides with the Kobayashi pseudodistance: $d_M = \bar{d}_M$.*

Proof. The inequality $\bar{d}_M \leq d_M$ is obvious because $F_M(v) = \inf |\xi|$ where the infimum is taken over all pseudoholomorphic mappings $f : \mathbf{D} \rightarrow M$, $f_*\xi = v$, and the norm is induced by the Poincaré metric. Let's prove the inverse. We follow the Royden's proof ([Ro]).

Let γ be such a smooth path from a point p to a point q that $\int_{\gamma} F_M < \bar{d}_M(p, q) + \varepsilon$. Due to upper semicontinuity there exists a continuous function h on the interval $[0, 1]$ such that $h(t) > F_M(\dot{\gamma}(t))$ and

$$\int_0^1 h(t) dt < \bar{d}_M(p, q) + \varepsilon,$$

i.e. for a sufficiently dense partition $0 = t_0 < t_1 < \dots < t_k = 1$ we have:

$$\sum_{i=1}^k h(t_{i-1})(t_i - t_{i-1}) < \bar{d}_M(p, q) + \varepsilon.$$

Let us consider arbitrary pseudoholomorphic curve $u_t^\gamma : \mathbf{D}_\delta \rightarrow M$ which satisfies the conditions $u_t^\gamma(0) = \gamma(t)$ and $(u_t^\gamma)_*e = \dot{\gamma}(t)$. Define for small $\Delta t \in \mathbb{R}_+ \subset \mathbb{C}$ the curve $\hat{\gamma}(t; \Delta t) = u_t^\gamma(\Delta t)$. Since $\hat{\gamma}(t; \Delta t) = \gamma(t + \Delta t) + O(|\Delta t|^2)$ then due to proposition 2 for small enough Δt we have:

$$\begin{aligned} d_M(\gamma(t), \gamma(t + \Delta t)) &\leq d_M(\gamma(t), \hat{\gamma}(t; \Delta t)) + d_M(\hat{\gamma}(t; \Delta t), \gamma(t + \Delta t)) \\ &\leq F_M(\dot{\gamma}(t))\Delta t + O(|\Delta t|^2) \leq (1 + \varepsilon)h(t)\Delta t. \end{aligned}$$

Thus for sufficiently dense partition of the interval $[0, 1]$ we have:

$$\begin{aligned} d_M(p, q) &\leq \sum_{i=1}^k d_M(\gamma(t_{i-1}), \gamma(t_i)) \\ &\leq (1 + \varepsilon) \sum_{i=1}^k h(t_{i-1})(t_i - t_{i-1}) < (1 + \varepsilon)(\bar{d}_M(p, q) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we are done. □

Chapter 3

Connection with h -principle

Let us call a pseudoholomorphic disks fiber bundle any embedding (immersion) $\Phi : \mathbf{D}_R \times N^{2n-2} \rightarrow M$ such that all the mappings $\Phi|_{D_R \times \{x\}}$ are pseudoholomorphic.

Proposition 5. *Let (M, J) be a 4-dimensional almost complex manifold. For any embedded (immersed) pseudoholomorphic disk $f : \mathbf{D}_R \rightarrow M$ and any $\varepsilon > 0$ a small neighborhood of the image $f(\mathbf{D}_{R-\varepsilon})$ can be fibered by pseudoholomorphic disks.*

Proof. We can change the almost complex structure J in a neighborhood of the boundary of the disk $f(\partial\mathbf{D}_R)$ turning it into the standard complex. After this we can reduce the manifold M gluing up the disk into the sphere. Our manifold takes the form $S_R^2 \times N^2$ (the subscript for the sphere has the following meaning: for our future purposes we introduce a symplectic form ω taming J , $\omega(\xi, J\xi) > 0$, of the form $\omega = \omega_1 \oplus \omega_2$, the volume being $\text{vol}_{\omega_1}(S_R^2) = \pi R^2$). We may also change the almost complex structure everywhere outside a small neighborhood of the disk image turning it into the standard integrable. After the corresponding gluing up we may assume that our manifold takes the form $M^4 = S_R^2 \times S_{2R}^2$ and in addition the almost complex structure J is the standard complex out of a small neighborhood of $f(\mathbf{D}_{R-\varepsilon})$. Denote by $A \in H_2(M; \mathbb{Z})$ the homological class of the sphere $S_R^2 \times \{*\} \subset M$. According to [Gr1] for an almost complex structure J of a general position there exists a unique pseudoholomorphic A -sphere through any point of the manifold M . These spheres form in totality a pseudoholomorphic

disks fiber bundle. The absence of intersections follows from nonnegativity of the intersection number ([Gr1]), this construction is similar to lemmas 3.5 and 4.1 from [M1] where the smoothness of the fibration is also proved. The intersection of the obtained bundle with a small neighborhood of the image $f(\mathbf{D}_{R-\varepsilon})$ gives us a fibration of this image (with the initial almost complex structure J) by pseudoholomorphic disks. In order to get rid of the general position condition for the structure J on $S_R^2 \times S_{2R}^2$ we perturb it and then go to limit with the perturbation tending to zero and use the compactness theorem ([MS]). \square

According to [Gau] (cf. [Kr]) for any almost complex (M^{2n}, J) there exists a canonical almost complex structure $J_{[1]}$ on the corresponding manifold of pseudoholomorphic 1-jets $J_{PH}^1(\mathbf{D}_R; M)$ such that the canonical projection $\pi : J_{PH}^1(\mathbf{D}_R; M) \rightarrow M$ is a pseudoholomorphic mapping. Any pseudoholomorphic mapping $f : \mathbf{D}_R \rightarrow M$ can be canonically lifted to a pseudoholomorphic mapping $j^1 f : \mathbf{D}_R \rightarrow J_{PH}^1(\mathbf{D}_R; M)$.

In any even dimension $2n$ similarly to proposition 5 one can prove that for any embedded (immersed) pseudoholomorphic disc $f : \mathbf{D}_R \rightarrow M^{2n}$ and any sufficiently close to its image point there exists a pseudoholomorphic disk $f' : \mathbf{D}_{R-\varepsilon} \rightarrow M^{2n}$ through this point. We may consider such a map $g' : \mathbf{D}_{R-\varepsilon} \rightarrow J_{PH}^1(\mathbf{D}_{R-\varepsilon}; M)$ constructed by the map $g = j^1 f$. Note that the constructed map is not necessary the jet lift of some map, $g' \neq j^1(\pi \circ g')$. If the equality $g' = j^1 f'$ holds then, in accordance with the terminology from [Gr2], the corresponding map should be called *holonomic*. Closeness of the initial points of the disks g and g' (i.e. the points $g(0)$ and $g'(0)$) in $J_{PH}^1(\mathbf{D}_{R-\varepsilon}; M)$ means closeness of initial points and initial directions for the maps f and f' in T_*M . Thus proposition 3 implies

Theorem 2. *There exists an embedded (immersed) holonomic pseudoholomorphic disk $g' : \mathbf{D}_{R-\varepsilon} \rightarrow J_{PH}^1(\mathbf{D}_{R-\varepsilon}; M)$ through every point sufficiently close to the image of an embedded (immersed) holonomic pseudoholomorphic disk $g : \mathbf{D}_R \rightarrow J_{PH}^1(\mathbf{D}_R; M)$.* \square

We may assume we have a family of pseudoholomorphic disks $g_\alpha : \mathbf{D}_{R-\varepsilon} \rightarrow J_{PH}^1(\mathbf{D}_{R-\varepsilon}; M)$ which fill out a neighborhood of the image of $g_0 = j^1 f$. Under which conditions this family may be assumed fibering (as in proposition 5)?

Chapter 4

Hyperbolicity and nonhyperbolicity

Almost complex manifold (M, J) is called *hyperbolic* if the pseudodistance d_M is a distance. Let us consider arbitrary norm $|\cdot|$ on the tangent bundle $\tau_M : TM \rightarrow M$ and let us consider the corresponding bundle of unit tangent vectors $\tau_M^{(1)} : T_1M \rightarrow M$. Consider the restriction of the Kobayashi-Royden pseudonorm on this submanifold, $F_M^{(1)} : T_1M \rightarrow \mathbb{R}$. Proposition 2(ii) and theorem 2(i) from [KO] imply

Theorem 3. *The function $F_M^{(1)}$ is bounded on compact subsets in M . The manifold M is hyperbolic if and only if $F_M^{(1)} \neq 0$ and the function $(F_M^{(1)})^{-1}$ is bounded on compact subsets in M , i.e. $F_M^{(1)}$ is bounded away from zero on compact subsets.* \square

Note that theorems 1 and 3 imply theorem 2(ii) from [KO], that is the almost complex version of Brody's theorem which states that for compact (possibly with boundary) manifold M hyperbolicity is equivalent to the absence of entire pseudoholomorphic curves, i.e. nontrivial pseudoholomorphic mappings $f : \mathbb{C} \rightarrow M$.

Now let us consider the case of nonhyperbolic manifold M , for example let it possess pseudoholomorphic spheres. In the case of general position for the almost complex structure J , which is tamed by some symplectic form ω on M , the set of all pseudoholomorphic spheres in a fixed homology class $A \in H_2(M; \mathbb{Z})$ (completed for compactness by the set of decomposable rational

curves) is a finite-dimensional manifold $\mathcal{M}(A; J)$ ([Gr1, MS]). We define by the *reduction procedure* some pseudodistance on this manifold. Namely for any two pseudoholomorphic spheres $f_i : S^2 \rightarrow M$ defined up to holomorphical reparametrization of S^2 let

$$d_{\mathcal{M}}([f_1], [f_2]) = d_M(p_1, p_2), \quad (\dagger)$$

where $p_i \in \text{Im}(f_i)$ are arbitrary points on the images.

As example note that defined pseudodistance is a distance for the manifold $M^4 = \Sigma_g^2 \times S^2$ with $g > 1$ with some almost complex structure. Actually by methods of [Gr1], as in proposition 5, one may prove that M is fibered by pseudoholomorphic spheres and $\mathcal{M} \simeq \Sigma_g^2$.

Note that it is shown in the paper [M2] that for N large enough the manifold $\mathcal{M} \times \mathbb{R}^{2N}$ possesses a homotopically canonical almost complex structure \tilde{J} . Applying the reduction procedure we may assume that the Kobayashi pseudodistance $d_{\mathcal{M} \times \mathbb{R}^{2N}}$ induces a pseudodistance on \mathcal{M} . There is an existence question for almost complex structures \tilde{J} such that the induced pseudodistance on \mathcal{M} equals to the reduced pseudodistance $d_{\mathcal{M}}$ defined by the formula (\dagger) .

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